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Dynamic Slope Scaling Procedure and Lagrangian Relaxation with Subproblem Approximation

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Abstract. The dynamic slope scaling procedure (DSSP) is an efficient heuristic algorithm that provides good solutions to the fixed-charge transportation or network flow problem. However, the procedure is graphically motivated and appears unrelated to other optimization techniques. In this paper, we formulate the fixed-charge problem as a mathematical program with complementarity constraints (MPCC) and show that DSSP is equivalent to solving MPCC using Lagrangian relaxation with subproblem approximation.

Key words: complementarity, fixed-charge transportation problem, lagrangian relaxation, MPEC, network flows.

1. Introduction

Consider an optimization problem of the form:

$$\begin{aligned} \text{FC:}\quad & \min \sum_{j=1}^{n} f_j(x_j) \\ & \text{s.t.}\quad Ax = b, \\ & \qquad\quad x_j \geqslant 0, \qquad \forall j = 1, \ldots, n, \end{aligned}$$

where

$$f_j(x_j) = \begin{cases} 0 & \text{if } x_j = 0, \\ s_j + c_j x_j & \text{if } x_j > 0 \end{cases}$$

and $c_j \geqslant 0$ for each j. Although it is more general to assume that $s = (s_1, \ldots, s_n)^T \geqslant 0$ and $s_j > 0$ for some j, we assume for simplicity that $s_j > 0$ for all j. (When $s = 0$, FC reduces to a linear program, a problem that can be solved efficiently by the simplex algorithm.) When A is a node-arc incidence matrix of a network and b is a vector of valid exogenous amounts of flows into and out of each node, FC is known in the literature as the fixed-charge transportation or network flow problem (see, e.g., Balinski, 1961; Kuhn and Baumol, 1962; Murty, 1968) and has applications in, e.g., network design, plant location, and production scheduling. Many have proposed exact (see, e.g., Cabot and Erenguc, 1984; Palekar et al., 1990; Lamar and Wallace, 1997) and heuristic (see, e.g., Diaby, 1991, Khang and Fujiwara, 1991; Kim and Pardalos, 1999) algorithms to solve the problem.

Among the heuristics, the dynamic slope scaling procedure (DSSP) proposed by Kim and Pardalos (1999) works well in practice. Since its introduction, there have been several extensions of DSSP to solve, e.g., piecewise linear network flow problems (see, Kim and Pardalos, 2000a, b) and the fixed-charge multi-commodity network flow problem (see, Eksioglu et al., 2002). Several (e.g., Bai et al., 2003) have also used it in other applications.

On the surface, DSSP appears unrelated to other optimization techniques and its principal idea is graphically motivated. Our goal in this paper is to establish relationships between DSSP and Lagrangian relaxation. More specifically, we formulate FC as a mathematical program with complementary constraints (MPCC) and show that DSSP is equivalent to solving MPCC using a version of Lagrangian relaxation that solves the subproblem approximately and uses Karush–Kuhn–Tucker (KKT) multipliers from MPCC instead of subgradients to find improved solutions. In nonlinear programming, the existence of KKT multipliers at a given point typically indicates that it is either locally or globally optimal when some constraint qualifications hold. In our case, the MPCC formulation of the fixed-charge problem does not satisfy the Mangasarian-Fromovitz constraint qualification (MFCQ) (see, e.g., Luo et al., 1997). This renders the set of KKT multipliers unbounded at points that are neither local nor global optimal to MPCC (see Gauvin, 1977). However, some KKT multipliers at non-optimal points do provide information that leads to improved solutions (see, e.g., Fletcher et al., 2002; Fletcher and Leyffer, 2002).

For the remainder, Section 2 reviews a version of DSSP for FC and Section 3 formulates the fixed-charge problem as a MPCC and discusses its properties. Section 4 presents a Lagrangian relaxation technique for MPCC and shows that the algorithm is equivalent to DSSP.

2. Dynamic Slope Scaling Procedure

For reference in subsequent sections, we state a version of DSSP for FC. (See Kim and Pardalos, 1999, for other versions.) To motivate DSSP, Kim and Pardalos (1999) observe that the objective function of FC is concave. Thus, there must exist an extreme point of the feasible region that is optimal to FC (see, e.g., Bazaraa et al., 1993). Furthermore, such an extreme point must be optimal to a linear program of the form: min $\{\pi^T x : Ax = b, x_j \geqslant 0,\ j = 1, \ldots, n\}$ for some cost vector π. To find the correct π, DSSP solves the linear program with an initial cost vector π^0 and uses an optimal solution to revise π^0 and obtain an updated cost vector π^1. Then, DSSP

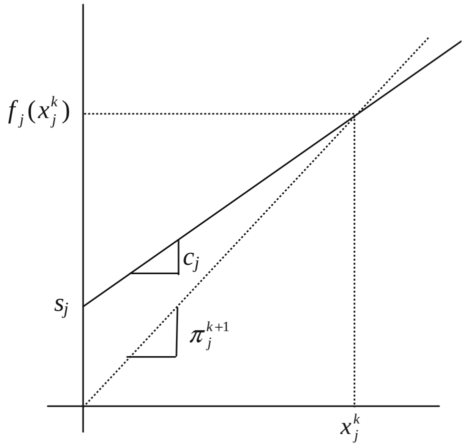

Figure 1. The slope calculation in Step 2 of DSSP.

resolve the linear program with π^1 and again uses an optimal solution to revise π^1 and obtain an updated cost vector π^2. This process continues until the difference between optimal solutions to two consecutive linear programs is sufficiently small. Mathematically, DSSP can be stated as follows:

Dynamic Slope Scaling Procedure

Step 0: Set $x^0=0$, $k = 1$, and $\pi_j^1=c_j$, for $j=1,\ldots, n$. Go to Step 1.
Step 1: Let $x^k=$ argmin $\{x^T\pi^k\colon Ax=b,\ x_j \geqslant 0, \forall j\}$. If $\left\|x^k-x^{k-1}\right\| \leqslant \varepsilon$, stop and x^k is an approximate solution to FC. Otherwise, go to Step 2.
Step 2: Set $\pi_j^{k+1}=\begin{cases} c_j+s_j\Big/x_j^k & \text{if } x_j^k>0 \\ \pi_j^k & \text{if } x_j^k=0 \end{cases}$ and $k=k+1$. Go to Step 1.

When $x_j^k>0$, the update choice of π_j^{k+1} in Step 2 is the slope of the line that passes through two points, the origin and $\left(x_j^k,\ f_j(x_j^k)\right)$ (see Figure 1).

Otherwise, the new slope π_j^{k+1} is the same as the current slope, π_j^k. If $x_j^\ell=0$ for all $\ell \leqslant k$, this update choice implies that π_j^{k+1} equals c_j, the initial slope. On the other hand, when $x_j^\ell>0$ for some $\ell<k$, π_j^{k+1} is the slope from the most recent iteration in which $x_j^\ell>0$ for some $\ell<k$.

3. Mathematical Program with Complementarity Constraints

Let y_j be a binary (decision) variable indicating whether x_j is allowed to be positive. Then, FC is equivalent to the following mathematical program:

$$\begin{aligned}\text{MPCC}: \min\ & \sum_{j=1}^{n} y_j\left(s_j + c_j x_j\right) \\ \text{s.t.}\ & Ax = b, \\ & x_j \geqslant 0, && \forall j = 1, \ldots, n, \\ & 0 \leqslant y_j \leqslant 1, && \forall j = 1, \ldots, n, \\ & x_j(1 - y_j) = 0, && \forall j = 1, \ldots, n.\end{aligned}$$

For each j, the constraint $x_j(1-y_j)=0$ ensures that y_j is binary. When $x_j > 0$, y_j must equal one to make the expression $x_j(1-y_j)$ equal zero. Similarly, when $x_j = 0, y_j$ must be zero to minimize the objective function because $s_j > 0$. The theorem below shows that MPCC and FC are equivalent.

THEOREM 3.1. *The FC and MPCC are equivalent, in that an optimal solution to one problem is also optimal to the other.*

Proof. Let x^* and (x', y') be an optimal solution of FC and MPCC, respectively. Because x^* is optimal to FC, the following must hold.

$$\begin{aligned}\sum_{j=1}^{n} f_j(x_j^*) = \sum_{j:x_j^*>0} s_j + c_j x_j^* \leqslant \sum_{j:x_j'>0} s_j + c_j x_j' \\ = \sum_{j=1}^{n} y_j'(s_j + c_j x_j'),\end{aligned} \tag{3.1}$$

where the inequality follows because x' is feasible to FC. On the other hand, the following must also hold because (x', y') is optimal to MPCC.

$$\begin{aligned}\sum_{j=1}^{n} y_j'(s_j + c_j x_j') \leqslant \sum_{j=1}^{n} y_j^*(s_j + c_j x_j^*) = \sum_{j:x_j^*>0} s_j + c_j x_j^* \\ = \sum_{j=1}^{n} f_j(x_j^*),\end{aligned} \tag{3.2}$$

where $y_j^* = 1$ if $x_j^* > 0$. Then (3.1) and (3.2) imply that $\sum_{j=1}^{n} f_j(x_j^*) = \sum_{j=1}^{n} y_j'\left(s_j + c_j x_j'\right)$. Thus, FC and MPCC are equivalent. □

To characterize a class of solutions to MPCC, consider the following *restricted minimum cost flow problem* associated with a binary vector y:

$$
\begin{aligned}
\mathrm{RF}[y]\colon \min \ & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \ & Ax = b, \\
& x_j \geqslant 0, \ \ \forall j : y_j = 1, \\
& x_j = 0, \ \ \forall j : y_j = 0.
\end{aligned}
$$

We refer to an optimal solution, x, of RF[y] as an *interior point* solution, if $x_j > 0$ for every j such that $y_j = 1$. Observe that, if (x^*, y^*) solves MPCC, then x^* must be an *interior point* solution to RF[y^*]. To verify, assume that x^* is not an interior point solution to RF[y^*]. Thus, there must exists an index j such that $x_j^* = 0$ and $y_j^* = 1$. By setting $\bar{y}_j = 1$ when $x_j^* > 0$ and $\bar{y}_j = 0$, otherwise, the point $(x^*, \bar{y})$ is feasible and has a smaller objective value because $s_j > 0$. This contradicts the fact that (x^*, y^*) solves MPCC. So, x^* must be an interior point solution to RF[y^*]. In addition, we refer to any (x, y) feasible to MPCC as a *candidate* solution if x is an interior point solution to RF[y].

Below are two properties of MPCC relevant to the subsequent sections. Theorem 3.2 shows that no candidate solution satisfies the MFCQ. Furthermore, failing to satisfy MFCQ implies that the set of feasible KKT multipliers at every candidate soluton is either empty or unbounded (see Gauvin, 1977). For MPCC, Theorem 3.3 shows that the latter holds at every candidate solution.

THEOREM 3.2. *No candidate solution to MPCC satisfies MFCQ.*

Proof. Recall that MFCQ at a given point (x, y) requires that there exists a vector (d, δ) such that

$$
\begin{aligned}
& Ad = 0, \\
& d_j > 0 \qquad \text{if } x_j = 0, && (3.3) \\
& \delta_j > 0 \qquad \text{if } y_j = 0, \\
& \delta_j < 0 \qquad \text{if } y_j = 1, && (3.4) \\
& (1 - y_j) d_j - x_j \delta_j = 0, \quad \text{for all } j. && (3.5)
\end{aligned}
$$

Consider a candidate solution (x, y). For each j such that $y_j = 0$ and $x_j = 0$, (3.3) and (3.5) reduce to $d_j > 0$ and $d_j = 0$, respectively. These two conditions are contradictory and MFCQ does not hold. Similarly, for each j such that $y_j = 1, x_j$ must be positive because (x, y) is a candidate solution. Then, (3.4) and (3.5) imply that $\delta_j < 0$ and $x_j \delta_j = 0$, respectively. Again, these conditions are contradictory because $x_j > 0$. Thus, MFCQ does not hold. □

THEOREM 3.3. *The set of feasible KKT multipliers at every candidate solution is unbounded.*

Proof. Let (x, y) denote a candidate solution MPCC. First, consider the KKT conditions for RF[y]. Because x solves RF[y], there exists a pair of KKT multipliers $(\bar{\rho}, \bar{\lambda})$ such that

$$\begin{array}{ll} c_j - a_j^T\bar{\rho} - \bar{\lambda}_j = 0, & \forall j, \\ x_j\bar{\lambda}_j = 0, & \forall j, \\ \bar{\lambda}_j \geqslant 0, & \forall j : y_j = 1, \\ \bar{\lambda}_j \text{ unrestricted}, & \forall j : y_j = 0, \end{array}$$

where a_j represents the jth column of A.

Consider next the following KKT conditions for MPCC at (x, y):

$$c_j y_j - a_j^T\rho - \lambda_j + \xi_j(1 - y_j) = 0, \quad \forall j = 1, \ldots, n. \tag{3.6}$$

$$s_j + c_j x_j - \alpha_j + \varphi_j - \xi_j x_j = 0, \quad \forall j = 1, \ldots, n. \tag{3.7}$$

$$x_j\lambda_j = 0, \quad \forall j = 1, \ldots, n. \tag{3.8}$$

$$\alpha_j y_j = 0, \quad \forall j = 1, \ldots, n. \tag{3.9}$$

$$\varphi_j(y_j - 1) = 0, \quad \forall j = 1, \ldots, n. \tag{3.10}$$

$$\lambda_j, \alpha_j, \varphi_j \geqslant 0, \quad \forall j = 1, \ldots, n.$$

At any candidate solution (x, y), there are two cases to consider: $(x_j > 0, y_j = 1)$ and $(x_j = 0, y_j = 0)$.

When $x_j > 0$ and $y_j = 1$, (3.8) and (3.9) imply that $\lambda_j = 0$ and $\alpha_j = 0$. Consequently, (3.6) reduces to $c_j - a_j^T\rho = 0$ and setting $\rho = \bar{\rho}$, the multiplier from RF[y], ensures that (3.6) holds. In addition (3.7) becomes

$$s_j + c_j x_j + \varphi_j - \xi_j x_j = 0 \quad \text{or } \xi_j = \frac{s_j + c_j x_j + \varphi_j}{x_j} = c_j + \frac{s_j}{x_j} + \frac{\varphi_j}{x_j}.$$

Thus (3.7) and (3.10) hold for any $\varphi_j \geqslant 0$. Moreover, ξ_j can be made arbitrarily large by choosing $\varphi_j \geqslant 0$ arbitrarily large.

When $x_j = 0$ and $y_j = 0$, (3.8) and (3.9) holds automatically and (3.10) implies that $\varphi_j = 0$. Consequently, (3.6) and (3.7) reduce to $-a_j^T\rho - \lambda_j + \xi_j = 0$ and $\alpha_j = s_j$, respectively. The latter implies that α_j is positive, thereby satisfying the nonnegative requirement. For the former, letting $\rho = \bar{\rho}$ and $\xi_j = \lambda_j + a_j^T\bar{\rho}$, for any $\lambda_j \geqslant 0$, ensures that (3.6) holds. As before, the multiplier ξ_j can be made arbitrarily large by choosing $\lambda_j \geqslant 0$ arbitrarily large. In both cases, there are multipliers that can be made arbitrarily large, i.e., the set of feasible of KKT multipliers is unbounded. □

From the above proof, the following are feasible KKT multipliers for MPCC at a candidate solution (x, y):

$$\rho = \bar{\rho} \tag{3.11}$$

$$\lambda_j = \begin{cases} 0 & \text{if } x_j > 0, \\ \geqslant 0 & \text{if } x_j = 0, \end{cases} \tag{3.12}$$

$$\alpha_j = \begin{cases} 0 & \text{if } y_j = 1, \\ s_j & \text{if } y_j = 0, \end{cases}$$

$$\varphi_j = \begin{cases} \geqslant 0 & \text{if } y_j = 1, \\ 0 & \text{if } y_j = 0, \end{cases}$$

$$\xi_j = \begin{cases} c_j + \frac{s_j}{x_j} + \frac{\varphi_j}{x_j} & \text{if } x_j > 0, \\ \lambda_j + a_j^T \bar{\rho}_j & \text{if } x_j = 0, \end{cases} \tag{3.13}$$

where $\bar{\rho}$ is from a pair of feasible multipliers $(\bar{\rho}, \bar{\lambda})$ for RF[y].

4. Lagrangian Relaxation

The Lagrangian dual problem associated with MPCC is to maximize $L(\xi)$, where

$$L(\xi) = \min\left\{ \sum_{j=1}^{n} y_j(s_j + c_j x_j) + \xi_j x_j(1 - y_j) : Ax = b, x_j \geqslant 0, 0 \leqslant y_j \leqslant 1, \forall j \right\}.$$

One approach for maximizing $L(\xi)$ is via Lagrangian relaxation, a version of which is stated below:

Lagrangian Relaxation

Step 0: Choose $\xi^1 \geqslant 0$ and set $k = 1$.
Step 1: Let (y^k, x^k) solve the following subproblem.

$$L(\xi^k) = \min\left\{ \sum_{j=1}^{n} y_j(s_j + c_j x_j) + \xi_j^k x_j(1 - y_j) : Ax = b, x_j \geqslant 0, 0 \leqslant y_j \leqslant 1, \forall j \right\}.$$

Step 2: If $||x^k - x^{k-1}|| \leqslant \varepsilon$, stop. Otherwise, choose a new ξ^{k+1} and set $k = k+1$. Go to Step 1.

In Step 0, the constraint $x_j(1 - y_j) = 0$ in MPCC can be replaced by $x_j(1 - y_j) \leqslant 0$ because both x_j and $(1 - y_j)$ are nonnegative. Therefore, we can choose a nonnegative ξ^1.

The subproblem in Step 1 evaluates the (Lagrangian) dual function, $L(\cdot)$, at the point ξ^k. Moreover, the subproblem is a disjoint bilinear programming problem (see, e.g., Audet et al., 1999) that has an equivalent concave minimization formulation (see, e.g., Benson, 1985; Thieu, 1988). Several cutting plane and branch-and-bound algorithms (see, e.g., Konno, 1976; Gallo and Ulkucu, 1977; Vaish and Shetty, 1977; Al-Khayyal and Falk, 1983) globally solve the bilinear problem in a finite number of iterations.

To relate Lagrangian relaxation to DSSP, consider the following heuristic approach for the subproblem in Step 1. Initially, set $y_j = 0$ for all j. Doing so reduces the subproblem in Step 1 to the following (restricted) subproblem:

$$x^k = \arg\min \left\{ \sum_{j=1}^{n} \xi_j^k x_j : Ax = b, x_j \geqslant 0, \forall j \right\}.$$

Given x^k, construct a candidate solution to MPCC by setting $y_j^k = 1$, if $x_j^k = 1$, and $y_j^k = 0$ otherwise. This yields a candidate solution (x^k, y^k) feasible to MPCC.

In Step 2 of Lagrangian relaxation, ξ^{k+1} is typically chosen to be $\xi^k + \theta_k \tau^k$, where $\tau^k \in \partial L(\xi^k)$ and $\theta_k \geqslant 0$. However, if we use the above heuristic, this is not possible because $(x^k,\ y^k)$ does not necessarily solve the subproblem in Step 1 optimally and, therefore, does not provide information about the subdifferential $\partial L(\xi^k)$. On the other hand, $(x^k,\ y^k)$ is a candidate solution to MPCC and Theorem 3.3 indicates that the KKT multipliers exist. In the MPCC literature, several (e.g., Fletcher and Leyffer, 2002; Fletcher et al., 2002) have used these multipliers in conjunction with an SQP approach to successful solves many practical MPCC problems in Leyffer (2000). For our case, we can choose ξ^{k+1} from the multipliers associated with the candidate solution $(x^k,\ y^k)$. From (3.13), let $\xi_j^{k+1} = c_j + s_j \big/ x_j^k$ for each j such that $x_j^k > 0$. This corresponds to setting $\varphi_j = 0$ and Fletcher et al. (2002) refer to this choice of ξ^{k+1} as 'basic.' For the case where $x_j^k = 0$, let $(\lambda^k,\ \rho^k)$ denote the KKT multipliers associated with the restricted subproblem. Then, $\xi_j^k - a_j^T \rho^k - \lambda_j^k = 0$ or $\xi_j^k = a_j^T \rho^k + \lambda_j^k$. Thus, setting $\bar{\rho}_j = \rho_j^k$ and $\lambda_j = \lambda_j^k \geqslant 0$ when $x_j^k = 0$ satisfies (3.11) and (3.12), respectively. From (3.13), this choice of (ρ, λ) produces $\xi_j^{k+1} = a_j^T \rho^k + \lambda_j^k = \xi_j^k$ when $x_j^k = 0$.

Using the heuristic to solve the subproblem and the above choice of ξ^{k+1}, the Lagrangian relaxation becomes

Lagrangian Relaxation with Subproblem Approximation

Step 0: Choose $\xi^1 \geqslant 0$ and set $k = 1$.

Step 1: Let $x^k = \arg\min\left\{\sum_{j=1}^{n} \xi_j^k x_j : Ax = b, x_j \geqslant 0, \forall j\right\}$. If $\left\|x^k - x^{k-1}\right\| \leqslant \varepsilon$, stop and x^k is an approximate solution to MPCC. Otherwise, go to Step 2.

Step 2: Set $\xi_j^{k+1} = \begin{cases} c_j + s_j \big/ x_j^k & \text{if } x_j^k > 0 \\ \xi_j^k & \text{if } x_j^k = 0 \end{cases}$ and $k = k+1$. Go to Step 1.

However, the above algorithm is the same as DSSP.

When viewed in this manner, DSSP examines only candidate solutions. It uses the KKT multipliers from MPCC at the current candidate solution in an attempt to find an improved candidate solution by solving a corresponding Lagrangian subproblem approximately. Furthermore, this observation offers a more rigorous framework for the slope updating (or scaling) scheme that was originally motivated by Figure 1 and may explain the success of DSSP in solving the fixed-charge problem.

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